Filip Zaleski (1008241500) 21.01.2025

MAT336 Homework 1

1\. a.) Prove by induction that $|x_1 + x_2 + \dots + x_n| \leq |x_1| + |x_2| + \dots + |x_n|$

Let $n = 1$. Then, $|x_1| \leq |x_1|$ is trivially true.

Now let the inductive hypothesis be that the inequality holds for $n = k$:

$$|x_1 + x_2 + \dots + x_k| \leq |x_1| + |x_2| + \dots + |x_k|.$$

<u>Proposition</u>: The inequality holds for $n = k+1$;

$$|x_1 + x_2 + \dots + x_k| \leq |x_1| + |x_2| + \dots + |x_{k+1}|$$

<u>Proof</u>: The triangle inequality states $|A + B| \leq |A| + |B|$.

Setting $A = x_1 + x_2 + \dots + x_k$ and $B = x_{k+1}$, I get:

$$(\underbrace{x_1 + \dots + x_n}_{A} + \underbrace{x_{n+1}}_{B}) = \underbrace{(x_1 + \dots + x_n)}_{A} + \underbrace{x_{n+1}}_{B}$$

By the triangle inequality;

$$|x_1 + \dots + x_n + x_{n+1}| \leq |x_1 + \dots + x_n| + |x_{n+1}|$$

Now since I said that $|x_1 + \dots + x_n| \leq |x_1| + \dots + |x_n|$ in our inductive hypothesis, I can reexpress the above as:

$$|x_1 + \dots + x_{n+1}| \leq |x_1| + \dots + |x_n| + |x_{n+1}|$$

↳ In proving that the inductive hypothesis holds for $n+1$, I showed that the result holds for all integers $n \geq 1$.

b.) An infinite decimal $x = a_0.a_1a_2\dots$ is eventually periodic if there are positive integers n and k such that $a_{i+k} = a_i$ for all $i > n$. Show that any decimal expansion which is eventually periodic represents a rational number.

Suppose $x = a_0.a_1a_2a_3\dots$ becomes k-periodic after the n^{th} digit (i.e there exist positive integers n and k for which $a_{n+j+k} = a_{n+j}$ for all $j > 0$. I show that x is rational by shifting the decimal to begin right at the repeating block:

$$10^n x = a_0a_1\dots a_{n-1}.a_na_{n+1}a_{n+2}\dots$$

↳ Now the decimal expansion starts exactly where the repeating digits begin.

I multiply once again to obtain one repeat sequence to the left of the decimal:

$$10^k(10^n x) = 10^{n+k} x = \underbrace{a_0 a_1 \dots a_{n-1} a_n a_{n+1} \dots a_{n+k-1} . a_n a_{n+1} a_{n+2} \dots}$$

The remainder here is also the remainder for $10^n x$, so:

$10^{n+k} x - 10^n x$ gets rid of the repeating remainders. Thus the result is an integer that I can call c. Then, $10^{n+k} x - 10^n x = c$. Rearranging:

$$x = \frac{c}{10^{n+k} - 10^n}.$$

↳ Thus, since both c and $(10^{n+k} - 10^n)x$ are integers, x is a ratio of integers. In other words, x is rational.

c.) Show that the two formulations of the Archimedean property are equivalent.

Formulation 1 ⟶ Formulation 2:

Assume that for every $x, y > 0$, there is an $n \in \mathbb{N}$ such that $nx > y$.

Let $z > 0$ and take $x = z$ and $y = 1$. Thus, $\exists\, n \in \mathbb{N} \ni nz > 1$. This implies that $z > \frac{1}{n}$. Now I should be able to choose a k so that $10^{-k} < \frac{1}{n}$.

↳ Since $\frac{1}{n}$ is positive, I can indeed find k that satisfies this because 10^{-k} approximates zero as k grows. So, there has to be a $10^{-k} < \frac{1}{n}$.

Formulation 2 ⟶ Formulation 1:

Let $x, y > 0$. Set $z = \frac{y}{x}$, so $z > 0$.

By this formulation $\exists\, k \in \mathbb{Z} \ni 10^{-k} < z$. Thus, $10^{-k} x < y$.

↳ Since 10^{-k} can be replaced by $\frac{1}{n}$ for some large $n \in \mathbb{N}$, it must be the case that $nx > y$. So the first formulation is satisfied.

2. Describe an algorithm for adding two infinite decimals x and y, which uniquely defines an infinite decimal representing $x + y$, and which also gives the correct result when x and y are rational numbers.

Let $x = x_0 . x_1 x_2 x_3 \dots$ and $y = y_0 . y_1 y_2 y_3 \dots$ where x_0 and y_0 are integers and the rest are digits $\in \{0, \dots, 9\}$.

|| Add each index from left to right.
V

$S_0 = x_0 + y_0$, $S_1 = x_1 + y_1$, etc.

Since s_1 represents the first decimal place, is $s_1 \geq 10$, 1 carries backward and increases the integer sum.

↳ The same is true for s_2. If $s_2 = x_2 + y_2 \geq 10$, 1 is added to s_1. If this operation makes s_1 reach 10, 1 is carried backward to s_0, the integer part.

Note that once you have carried 1 into s_1 from s_2, s_1 is never again changed by future steps. That is, a carry from digit s_3 only goes as far as s_2, and so on.

↳ This procedure yields a unique infinite string of decimal digits:

$$s_0 . s_1 s_2 s_3 \ldots$$

Numerical example: Take $x = 12.956$, $y = 3.049$.

Integer part is $12 + 3 = 15$.

First decimal is $9 + 0 = 9$ → $15.9\ldots$

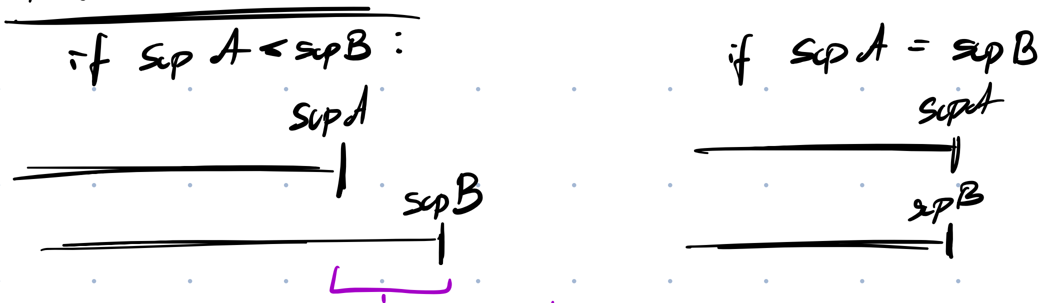
Second decimal is $5 + 4 = 9$ → $15.99\ldots$

Third decimal is $6 + 9 = 15$ ← there is a carry of one : 16.005

Up to the 1000^{th} decimal place, the sum is 16.005. We see in this example that each digit can be adjusted at most once. In an infinite decimal, each new carry affects only the most recently-computed digit, so no digit ever gets changed a second time.

3. Suppose that A and B are two nonempty subsets of the real numbers. Show that, if $\sup A < \sup B$, then there exists an element $b \in B$ that is an upper bound for A. Give an example to show that this is not always the case if we only assume $\sup A \leq \sup B$.

A visualization:



b can be anywhere here ; $\sup A \leq b \leq \sup B$

if $\sup A = \sup B$, there is no $b \in B > \sup A$.

A formal proof:

Let $A = \{a \in \mathbb{R} \mid 2 \leq a \leq 11\}$ and $B = \{b \in \mathbb{R} \mid 3 \leq b \leq 13\}$

In this case, $\sup A = 11$ and $\sup B = 13$ so $\sup A < \sup B$.

An upper bound for A is any $b > \sup A$.

↳ Since $\sup A < \sup B$, we can say that if $\sup A < b \leq \sup B$, b is a valid upper bound for A.

Applied to my example, then, we can pick any $b \in \mathbb{R}$ that meets $11 < b \leq 13$. This will remain an element of B since it is smaller than $\sup B$.

Note: no matter where $\inf B$ is, if $\sup A < \sup B$ then we know that there must be an arbitrarily small shared set $A \cap B \neq \emptyset \ni \{a \in A \mid a \in B\}$.

↳ In this case, $b = 12$ is both a part of B and an upper bound for A.

The case where $\sup A \leq \sup B$:

$\sup A \leq \sup B$ allows for the possibility that $\sup A = \sup B$.

Logically, there cannot exist any $b \in B$ that is different to $\sup A$ and still an upper bound. In fact the only element of B that is an upper bound in this case is the supremum of B itself.

4. A set B is said to be dense in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Prove why they are or are not. Suppose that $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

a.) The set of all rational numbers p/q with $q \leq 10$.

Call this $B_1 = \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \{1, 2, \dots, 10\}\right\}$.

↳ The smallest possible value of B_1 is $\frac{1}{10} = 0.1$. So, in any interval $0 < a, b < 0.1$, B_1 cannot appear. Thus, B_1 is not dense in $\mathbb{R}$.

b.) The set of all rational numbers p/q with q a power of 2.

Call this $B_2 = \left\{\frac{p}{2^n} : p \in \mathbb{Z}, n \in \mathbb{N}\right\}$

↳ The range of this is $0 \leq B_2 \leq \infty$. You can thus reach any arbitrarily small range in this set by taking a sufficiently large n to make the denominator huge and arrive at a tiny decimal expansion.

In other words, given $a < b$, you can chose n large enough that $\frac{1}{2^n} < (b-a)$.

You can then pick an integer p so that $\frac{p}{2^n}$ lies directly between a and b.

↳ Hence, B_2 is dense in $\mathbb{R}$.

c.) The set of all rational numbers $\frac{p}{q}$ with $10|p| \geq q$.

Call this $B_3 = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, q \leq 10|p|\}$

Let's inspect the smallest value of B_3, which occurs at the largest denominator $q = 10|p|$.

Then, $\frac{p}{q} = \frac{p}{10|p|} = \pm\frac{1}{10}$. So, $\left|\frac{p}{q}\right| \geq \frac{1}{10}$ for $p \neq 0$.

↳ We run into the same problem we did before, where there is a range around zero which this set is not a part of. Let's say that we took $a = -0.01$ and $b = 0.01$. B_3 can never reach any number $a < B_3 < b$.

Thus, B_3 is not dense.

5. Let D denote the set of all finite decimals.

a.) Let $x, y \in \mathbb{R}$. Prove that $x + y = \sup\{a+b : a, b \in D, a \leq x, b \leq y\}$.

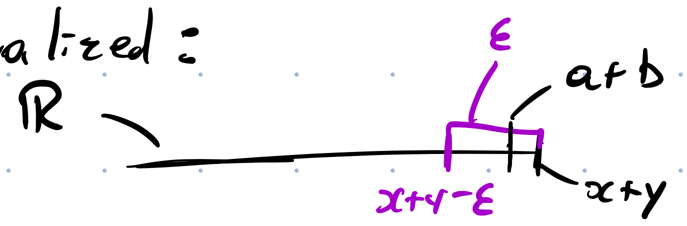
For any $a \in D_x$ and $b \in D_y$, we know $a \leq x$ and $b \leq y$. Thus, $a + b \leq x + y$.

Thus $x+y$ is an upper bound of $a+b$.

↳ To check whether this is a supremum, I show that:

for any $\varepsilon > 0$ $\exists\, a \in D_x, b \in D_y \ni a+b > x+y-\varepsilon$.

Visualized:



↳ Since a can approximate x arbitrarily closely from below (because $x = \sup D_x$) and b can approximate y ($y = \sup D_y$), I choose a with $a > x - \frac{\varepsilon}{2}$ and b with $b > y - \frac{\varepsilon}{2}$: $a + b > \left(x - \frac{\varepsilon}{2}\right) + \left(y - \frac{\varepsilon}{2}\right)$

This shows that $a+b$ can get arbitrarily close to $x+y$ from below, proving that $x+y$ is the supremum of the set $\{a+b : a \in D_x, b \in D_y\}$

b) Suppose that $x, y \in \mathbb{R}$ are positive. Show that:

$xy = \sup\{ab : a, b \in D, 0 \leq a \leq x, 0 \leq b \leq y\}$.

For any $a \in D$ and $b \in D$ s.t. $0 \leq a \leq x$ and $0 \leq b \leq y$, we know that $ab \leq xy$.

Hence, xy must be an upper bound for the set $\{ab\}$.

↳ To show whether xy is a supremum, I let $\varepsilon > 0$ and note that:

$\exists\, a \in D, b \in D, \ni a > x - \delta$ and $b > y - \delta$, where $\delta > 0$ is small enough to ensure $(x-\delta)(y-\delta) > xy - \varepsilon$. Then;

$ab > (x-\delta)(y-\delta) > xy - \varepsilon$.

↳ We can say that no number smaller than xy can be an upper bound of $\{ab\}$, proving that xy is the supremum.

c.) How do we define multiplication in general?

I propose a general precise definition of multiplication in $\mathbb{R}$:

For $x, y > 0$:

We already know that, for non-negative numbers;

$xy = \sup\{ab : a, b \in D, 0 \leq a \leq x, 0 \leq b \leq y\}$

For x or $y > 0$:

If $x > 0$ and $y < 0$, it is intuitive to state that $xy = -(|x| \cdot |y|)$

Where $|x| \cdot |y|$ is the product of their absolute values (trivially defined).

For both x and $y < 0$:

If $x < 0$ and $y < 0$, it is intuitive to state that $xy = |x| \cdot |y|$.

For one or both in $x, y = 0$:

Again, trivially true to say that $0y = 0$ and $0x = 0$.

↳ Thus, overall, I propose the general definition of multiplication for all real numbers $x, y \in \mathbb{R}$:

$$x \cdot y = \begin{cases} \sup\{ab : a, b \in D, 0 \leq a \leq |x|, 0 \leq b \leq |y|\}, & \text{if } x, y > 0 \\ -(|x| \cdot |y|), & \text{if } x < 0 \text{ and } y > 0 \text{ or viceversa} \\ |x| \cdot |y|, & \text{if } x, y < 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

6. For the following sets, find the supremum and infimum. Which have a max or min?

a.) $A = \{a + a^{-1} : a \in \mathbb{Q},\ a > 0\}$

$$a + a^{-1} \equiv a + \frac{1}{a} = (\sqrt{a})^2 + \left(\frac{1}{\sqrt{a}}\right)^2 = \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 + 2\left(\sqrt{a} \cdot \frac{1}{\sqrt{a}}\right)$$

Since $\sqrt{a} \cdot \frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{\sqrt{a}} = 1$, I can say $a + \frac{1}{a} = \left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 + 2$.

↳ Now since this term is raised to the power of 2, it's always positive:

$(\sqrt{a} - \frac{1}{\sqrt{a}})^2 \geq 0$. Thus: $a + \frac{1}{a} \geq 2$.

↳ $a + \frac{1}{a}$ is only $= 2$ when $a = 1$. Thus, inf $A = 2$ and the minimum of A is at 2 as well ($a=1$ is defined in the set).

b.) $B = \{a + (2a)^{-1} : a \in \mathbb{Q}, 0.1 \leq a \leq 5\}$

I'll treat this like a function and use calculus to analyse it's properties.

$f(a) = a + \frac{1}{2a}$.

↳ $f(0.1) = 0.1 + \frac{1}{2 \times 0.1} = 5.1$

↳ $f(5) = 5 + \frac{1}{2 \times 5} = 5 + 0.1 = 5.1$

To find the critical point, $f'(a) = 1 - \frac{1}{2a^2}$

↳ Setting $f'(a) = 0$ gives $1 - \frac{1}{2a^2}$ ∴ $2a^2 = 1$ ∴ $a = \frac{1}{\sqrt{2}}$

Since $\frac{1}{\sqrt{2}}$ lies in $[0.1, 5]$, it's valid.

$f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} + \frac{1}{2(\frac{1}{\sqrt{2}})} = \sqrt{2} \approx 1.414$.

↳ $1.414 < 5.1$, so this must be a local minimum.

↳ To determine whether this is an infimum and/or a minimum, I observe that:

- $f(\frac{1}{\sqrt{2}})$ does not satisfy the constraints defined in B because $\frac{1}{\sqrt{2}}$ is not rational. So there is no minimum of B.
- However, because $\mathbb{Q}$'s are dense, I can choose rational values of a that get arbitrarily close to $\frac{1}{\sqrt{2}}$, in turn making $f(a)$ arbitrarily close to what I said was its local minimum.

  ↳ Hence, inf $B = \sqrt{2}$.

→ To determine whether B has a supremum and/or a maximum, I note that:

- On both rational endpoints of it's range, 0.1 and 5, $f(a) = 5.1$.

↳ Thus, $\sup B = 5.1$.

- That this supremum is reachable (it's at rational a's) implicitly means that there is a maximum – so $\max B = 5.1$.

c.) $C = \{xe^{-x} : x \in \mathbb{R}\}$

I perform a similar analysis as in part b above.

Let $g(x) = xe^{-x}$, $x \in \mathbb{R}$. Then $g'(x) = e^{-x}(1-x)$.

$g'(x) = 0$: $e^{-x}(1-x) = 0$ $\therefore$ $e^{-x} = 0$ and $1-x=0$, in both cases $x = 1$.

↳ $g(1) = 1 \cdot e^{-1} = \frac{1}{e}$, a local maximum.

Behaviour as $x \to \pm \infty$:

- As $x \to \infty$, $e^{-x} \to 0$ so $xe^{-x} \to 0$ from above.
- As $x \to -\infty$, x remains negative but is positive, so $xe^{-x} \to -\infty$.

We know that the function can become arbitarily negative and it has a single peak at $x=1$. So:

- $\sup C = 1/e$, and since $x = 1 \in \mathbb{R}$, the constraint on C is met and this is a maximum.
- $\inf C \to -\infty$, so there is no infimum (implicitly, also no minimum).