C. Prove that if $a_n \leq b_n$ for $n \geq 1$, $L = \lim_{n \to \infty} a_n$, and $M = \lim_{n \to \infty} b_n$, then $L \leq M$.

F. (a) Let $x_n = \sqrt[n]{n} - 1$. Use the fact that $(1 + x_n)^n = n$ to show that $x_n^2 \leq 2/n$.
HINT: Use the Binomial Theorem and throw away most terms.

F. Let a, b be positive real numbers. Set $x_0 = a$ and $x_{n+1} = (x_n^{-1} + b)^{-1}$ for $n \geq 0$.

(a) Prove that x_n is monotone decreasing.
(b) Prove that the limit exists and find it.

4. Suppose that $\lim_{n \to \infty} a_n = L$. Show that

$$\lim_{n \to \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = L.$$

Is the converse true? Hint: Consider the sequence $a_n = (-1)^n$.

J. (a) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers. If $L = \liminf x_n$, show that there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\lim_{k \to \infty} x_{n_k} = L$.
(b) Similarly, prove that there is a subsequence $(x_{n_l})_{l=1}^{\infty}$ such that $\lim_{l \to \infty} x_{n_l} = \limsup x_n$.

B. Give a sequence (a_n) such that $\lim_{n \to \infty} |a_n - a_{n+1}| = 0$, but the sequence does not converge.

B. Show that $(0, 1)$ and $[0, 1]$ have the same cardinality as $\mathbb{R}$.

E. A real number α is called an **algebraic number** if there is a polynomial with integer coefficients with α as a root. Prove that the set of all algebraic numbers is countable.
HINT: First count the set of all polynomials with integer coefficients.

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MAT 336 Homework 2

1. Section 2.4, Exercise C

I use a proof by contradiction to illustrate that $L \leq M$ under the given conditions.

↳ I assume that $L > M$, the contrary. This allows me to say that $\varepsilon = \frac{L-M}{2} > 0$.

Now since $L = \lim_{n\to\infty} a_n$ and $M = \lim_{n\to\infty} b_n$, there must exist indices N_1 and N_2 such that for all $n \geq N_1$, $|a_n - L| < \varepsilon$ and for all $n \geq N_2$, $|b_n - M| < \varepsilon$.

Restated, for all $n \geq \max\{N_1, N_2\}$:

$$a_n > L - \varepsilon = \frac{L+M}{2} \quad \text{and} \quad b_n < M + \varepsilon = \frac{L+M}{2}.$$

↳ Thus, for sufficiently large n, $a_n > \frac{L+M}{2} > b_n$.

↳ But this is a direct contradiction of the given condition, "$a_n \leq b_n$ for all n."

Since the assumption $L > M$ led to a contradiction, we must have $L \leq M$.

2. Section 2.5, Exercise F

Part a.

I use the binomial theorem to expand:

$$(1+x_n)^n = 1 + \binom{n}{1}x_n + \binom{n}{2}x_n^2 + \binom{n}{3}x_n^3 + \dots$$

↳ the higher-order terms are positive because $x_n \geq 0$ for $n \geq 2$.

↓ Subtracting 1 from both sides

$$n - 1 = (1+x_n)^n - 1 = \left[\binom{n}{1}x_n + \binom{n}{2}x_n^2 + \dots\right] \geq \binom{n}{2}x_n^2 = \frac{n(n-1)}{2}x_n^2$$

Thus: $\frac{n(n-1)}{2}x_n^2 \leq n-1$

dividing by $\frac{n(n-1)}{2}$

$$x_n^2 \leq \frac{2(n-1)}{n(n-1)} = \frac{2}{n}$$

Part b.

We know from the above solution to part a. that $x_n = n^{1/n} - 1$ satisfies:

$x_n^2 < \frac{2}{n}$. Here as $n \to \infty$, $\frac{2}{n} \to 0$ (the numerator becomes enormous).

↳ Hence, $x_n^2 \to 0$, which necessitates $x_n \to 0$. Thus:

$n^{1/2} = 1 + x_n \to 1$ as $n \to \infty$, or:

$$\lim_{n \to \infty} n^{1/n} = 1.$$

3. Section 2.6, Exercise F

Part a.

$$\begin{cases} x_0 = a \\ x_{n+1} = (x_n^{-1} + b)^{-1} \text{ for } n \geq 0. \end{cases}$$

$$↳ (x_n^{-1} + b)^{-1} \equiv \frac{1}{(1/x^n + b)} = \frac{x_n}{1 + bx_n}.$$

We are also told that $b > 0$, and we know that $x_n > 0$.

We thus know that: $1 + bx_n > 1$. Then:

$$\left(\frac{x_n}{1+bx_n} < x_n \right) \xrightarrow{\text{implies}} x_{n+1} < x_n$$

↳ This is a demonstration that $\{x_n\}$ is strictly monotone decreasing.

Part b.

We should first think about what the limit could be.

$x_0 = a$, and we know that $a > 0$.

at $n=0$, $x_1 = x_0 / 1+bx_0$

at $n=1$, $x_2 = x_1 / 1+bx_1$

it seems that each consecutive term scales the preceding term by $1+b$.

We know that b is a positive number. Thus, it follows inductively that all x_n are positive. We also know, however, that $\{x_n\}$ is a strictly decreasing sequence. The combination of these 2 observations mandates that x_n must be bounded below by zero.

The monotone convergence theorem states that a decreasing sequence that is bounded has a limit.

↳ I call this limit L. Applying the limit to the sequence $x_{n+1} = \frac{1}{1/x_n + b}$ gives:

$$\lim_{n\to\infty} \frac{1}{1/x_n + b} = \frac{1}{1/L + b}.$$ Solving for L as is done in the textbook:

$$L\left(\frac{1}{L} + b\right) = 1 \quad \text{so} \quad \frac{L}{L} + Lb = 1 \quad \text{so} \quad 1 + bL = 1 \quad \text{so} \quad bL = 0,$$ thus

$L = 0.$ The limit being zero is in line with findings from part a, which outlined that all the terms of the sequence are positive while the sequence is strictly monotone decreasing.

4. Suppose that $\lim_{n\to\infty} a_n = L$. Show that $\lim_{n\to\infty} \frac{a_1 + a_2 + \dots + a_n}{n} = L$.
Is the converse true? Hint: consider the sequence $a_n = (-1)^n$.

I split the sum at some index N:

$$\frac{a_1 + a_2 + \dots + a_N}{n} = \underbrace{\frac{a_1 + \dots + a_N}{n}}_{\text{finite initial block}} + \underbrace{\frac{a_{N+1} + \dots + a_N}{n}}_{\text{tail}}$$

↳ Now because $a_1 + \dots + a_N$ is a fixed sum while n grows, the denominator explodes as $n \to \infty$:

$$\frac{a_1 + \dots + a_N}{n} \to 0 \text{ as } n \to \infty$$

Since $\lim_{k\to\infty} a_k = L$, we can choose an N so that for all $k \geq N$, a_k lies within ε of L. Hence, for $n \geq N$:

$$(L-\varepsilon)(n-N+1) \leq a_{N+1} + \dots + a_n \leq (L+\varepsilon)(n-N+1)$$

Dividing by n

$$\frac{(n-N+1)(L-\varepsilon)}{n} \leq \frac{a_{N+1} + \dots + a_n}{n} \leq \frac{(n-N+1)(L+\varepsilon)}{n}$$

As $n \to \infty$, the factor $\frac{n-N+1}{n} \to 1$, so the tail average condenses into $[L-\varepsilon, L+\varepsilon]$

2- This tells us that the limit is indeed L, because ε can be made arbitrarily small. So:

$$\lim_{n\to\infty} \frac{a_1 + \dots + a_n}{n} = L.$$

As to whether the converse is true, that is, if the averages converge does the sequence a_n have to converge to L;

I note from the provided hint that the sequence $a_n = (-1)^n$ does not converge, since it keeps jumping between $+1$ and -1. This is the case despite the fact that the average of the sequence $\frac{1}{n}\sum_{k=1}^{n} (-1)^n$ does converge.

Overall, the statement that "$\lim a_n = L$ implies $\lim \frac{1}{n}\sum a_k = L$" is true, but the converse fails.

5. Section 2.7, Exercise J.

Part a.

By the definition of a limit of the infimum of x_n, we let:

$$L = \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{m\geq n} x_m\right).$$

We then define the sequence $y_n = \inf_{m\geq n} x_m$.

↳ It represents the infimum of all the terms $\{x_n, x_{n+1}, \dots\}$.

By definition, then, $\lim_{n\to\infty} y_n = L$. As n increases, the infimum can only go up or stay the same. Hence, $y_1 \leq y_2 \leq y_3 \leq \dots$.

Applying the definition of infimum, for each k there must be some term x_m with $m \geq k$, whose value is at most $y_k + \frac{1}{k}$.

↳ That is, $x_m \leq y_k + \frac{1}{k}$ for all k where $m \geq k$.

We denote the choice of the above m by n_k, defining n_k so that it's strictly increasing in k (thereby making $\{x_{n_k}\}$ a valid subsequence):

$n_1 < n_2 < n_3 < \dots$, so that $n_k \geq k$.

Then, the constructed system allows us to state that, for each k;

$$\underbrace{y_k \leq y_{n_k}}_{1} \underbrace{\leq x_{n_k}}_{2} \underbrace{\leq y_{n_k} + \frac{1}{k}}_{3}$$

1. because we specified that $n_k \geq k$ (and thus the infimum increases)
2. because y_{n_k} is the infimum of all terms from the index n_k onward, so it cannot exceed any

particular index x_{n_k}.

③ because as per our construction at the beginning, $x_{n_k} \leq y_k + \frac{1}{k}$.

Now since $y_k \to L$ and $y_k + \frac{1}{k} \to L$, the inequalities $y_k \leq x_{n_k} \leq y_k + \frac{1}{k}$ can be interpreted via Squeeze Theorem to suggest that x_{n_k} also converges to L;

$$\lim_{k \to \infty} x_{n_k} = L.$$

↳ We have thus succeeded in demonstrating that there is a subsequence $\{x_{n_k}\}$ that converges to the limit of the infimum.

Part b.

A similar reasoning can be employed to tackle this question.

First, we define:

$$M = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{m \geq n} x_m\right).$$

We then construct z_n, the supremum alternative of y_n from the last question.

$$z_n = \sup_{m \geq n} x_m, \text{ and } \lim_{n\to\infty} z_n = M.$$

↳ It must be the case that $z_1 \geq z_2 \geq z_3 \geq \dots$, since the supremum can only decrease or stay the same as the index n grows.

Now for each n, since z_n is the least upper bound of $\{x_n, x_{n+1}, \dots\}$, we can pick some $l_n \geq n$ so that:

$$x_{l_n} > z_n - \frac{1}{n}.$$

↳ To construct a proper subsequence, we choose $l_1 < l_2 < \dots$.

We get the resulting system:

$$z_n - \frac{1}{n} < x_{l_n} \leq z_{l_n} \leq z_n.$$

From here, we can infer that since $l_n \geq n$, z_{l_n} is one of the 'tail' suprema – so $z_{l_n} \leq z_n$.

As $n \to \infty$, then, both z_n and $z_n - \frac{1}{n}$ converge to M. These surrounding x_{l_n} in the above inequality, we can apply the Squeeze Theorem again to state that $x_{l_n} \to M$.

↳ We have thus successfully constructed a subsequence that converges to $\limsup x_n$.

6. Section 2.P, Exercise 3.

Take $a_n = \sqrt{n}$. Then;

$$a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

Here, as $n \to \infty$, the denominator $\sqrt{n+1} + \sqrt{n}$ grows infinitely large. Thus:

$\lim_{n\to\infty} |\sqrt{n+1} - \sqrt{n}| = 0$ (the $+1$ becomes insignificant).

↳ This can be reexpressed as $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$.

Now for any real number A, one can pick an $n > A^2$ to make $\sqrt{n} > A$. Thus, $\{\sqrt{n}\}$ goes beyond any finite bound and does not converge.

7. Section 2.9, Exercises B and E.

Exercise B

To see that the intervals $[0,1]$ and $[1,0]$ are the same size, I demonstrate a 2-way bijection and apply the Cantor-Bernstein theorem.

One explicit bijection is the standard logistic map;

$$f : (0,1) \to \mathbb{R}, \text{ with } f(x) = \ln\left(\frac{x}{1-x}\right).$$

↳ For each $x \in (0,1)$, the ratio $\frac{x}{1-x}$ is a positive $\mathbb{R}$ - so $\ln\left(\frac{x}{1-x}\right)$ parades all positive real numbers as x increases in $(0,1)$.

↳ As $x \to 0^+$, $\frac{x}{1-x} \to 0^+$ and $\ln\left(\frac{x}{1-x}\right) \to -\infty$

↳ As $x \to 0^-$, $\frac{x}{1-x} \to +\infty$ and $\ln\left(\frac{x}{1-x}\right) \to +\infty$.

The above statements satisfy that f is a continuous bijection from $(0,1)$ onto $\mathbb{R}$. This in turn means that $(0,1)$ and $\mathbb{R}$ have the same cardinality.

Now, to show that $(0,1)$ has the same cardinality as $[0,1]$;

1) Consider that every x in $(0,1)$ is inherently also in $[0,1]$.

2) Consider also that, to construct the injection $[0,1] \to (0,1)$, we need a system that pushes the endpoints ever so slightly inside the interval. For instance:

$$h(x) = \begin{cases} \frac{1}{2}x & \text{for } 0 \leq x \leq \frac{1}{2}. \\ \frac{1}{2} + \frac{1}{x}(x-1) & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

↳ This function is one-to-one and it satisfies the interval $(0,1)$.

Now because we have injections both ways, the Cantor-Bernstein theorem mandates a bijection between $(0,1)$ and $[0,1]$. Hence, they share the same cardinality.

↳ In step 1, however, we showed that $(0,1)$ has the same cardinality as $\mathbb{R}$.
Therefore, all three sets $(0,1)$, $[0,1]$, and $\mathbb{R}$ are in one-to-one correspondence with each other, or:

$$|(0,1)| = |[0,1]| = |\mathbb{R}|.$$

Exercise E

A polynomial of n^{th} degree can be written as:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \text{ where each } a_i \in \mathbb{Z} \text{ and } a_n \neq 0.$$

↳ The integer coefficients can be written as: $(a_n, a_{n-1}, \dots, a_0)$, a finite tuple.
Now since each polynomial corresponds to a finite sequence, and the set of all finite sequences of integers is countable, we note that there are only countably many such polynomials.

Now suppose that α is a real root of some $P(x)$ of degree n. That polynomial may have up to n roots in $\mathbb{R}$, which can be listed in increasing order.
We can assign to α the extra information about which real root it is in the ordered list:

$p=1$ if α is the smallest real root, $p=2$ if it is the next consecutive root, etc.

↳ Then, α is encoded by $(a_n, a_{n-1}, \dots, a_0, p)$; again, a finite sequence of integers.

We know that different algebraic numbers get distinct representations. We thus have an injection from real algebraic sequences to finite algebraic sequences.

↳ Because we have embedded the algebraic numbers into a countable set (the set of all finite sequences of integers), it follows that there are at most countably many of them.

↳ Reexpressed, (the set of $\mathbb{R}$ algebraic numbers) $\subseteq$ (a countable set)

Hence the algebraic numbers form a countable subset of $\mathbb{R}$.