MAT336

Homework 3 Questions

1. Section 3.2, Exercises C, D, E, L.

C. Euler proposed that $1-2+4-8+\cdots=\sum_{n=0}^{\infty}(-2)^n=\frac{1}{1-(-2)}=\frac{1}{3}$. What is wrong with this?

D. Let $(a_n)_{n=1}^{\infty}$ be a monotone decreasing sequence of positive real numbers. Show that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

E. Apply Exercise D to the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ for $p>0$. For which values of p does this converge?

L. (a) Find a convergent series $\sum_{n=1}^{\infty} a_n$, with positive entries, such that $\lim_{n\to\infty}\sqrt[n]{a_n}=1$.

(b) Find a divergent series with the same property.

2.

Dirichlet's test for convergence says that, if the partial sums of the series $\sum_{k=1}^{\infty} x_k$ are bounded and if the sequence (y_k) is monotonically decreasing and converges to zero, then the series $\sum_{k=1}^{\infty} x_k y_k$ converges.

(a) Prove the Dirichlet test using the fact that

$$\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1}), \tag{1}$$

for all $n \geq 1$, where $s_n = \sum_{k=1}^{n} x_k$. You do not have to prove formula (1). Hint: Prove that $\sum_{k=1}^{\infty} s_k(y_k - y_{k+1})$ converges using the comparison test.

(b) Show that the alternating series test is a special case of Dirichlet's test.

3. Section 4.1, Exercises G, H

G. Let $\mathbf{x}$ and $\mathbf{y}$ be two nonzero vectors in $\mathbb{R}^2$, and let the angle between them be θ. Prove that $\langle \mathbf{x},\mathbf{y}\rangle = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta$. HINT: If $\mathbf{x}$ makes the angle α with the positive x-axis, then $\mathbf{x} = (\|\mathbf{x}\|\cos\alpha, \|\mathbf{x}\|\sin\alpha)$.

H. For nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, define θ by $\|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta = \langle \mathbf{x},\mathbf{y}\rangle$, and call this the angle between them.

(a) Prove the **cosine law**: If $\mathbf{x}$ and $\mathbf{y}$ are vectors and θ is the angle between them, then $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta + \|\mathbf{y}\|^2$.

(b) Prove that $\langle \mathbf{x},\mathbf{y}\rangle$ can be defined using only the norms of related vectors.

4.

(Bonus) Prove that the series

$$\sum_{n\in\mathbb{Z}} \frac{1}{x-n} = \frac{1}{x-1} + \frac{1}{x+1} + \frac{1}{x-2} + \frac{1}{x+2} + \cdots, \tag{2}$$

where $x \notin \mathbb{Z}$, is conditionally convergent. Hint: To show that the series (2) is conditionally convergent, combine terms to create a new series, which you can then show is convergent by the comparison test. To show that the series (2) is not absolutely convergent, use the comparison test again, but this time, compare the terms to those of a series which you know diverges.

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MAT336 Homework 3

1. Section 3.2

Exercise C

The problem with Euler's proposal is that the series
$\sum_{n=0}^{\infty}(-2)^n = 1-2+4-8+\dots$ does not converge.

We can see this by examining the partial sums $\sum_{k=1}^{n} s_n$:

$s_1 = 1$, $s_2 = 1-2=-1$, $s_3 = 1-2+4=3$, $s_4 = 1-2+4-8=-5$, $s_5 = 1-2+4-8+15=11$

↳ As visible, the sequence of partial sums oscillates between the positive and negative, and grows unboundedly. Since the sequence of partial sums does not converge, the series must diverge.

Euler arrived at his proposition by applying the formula for the sum of a geometric series:
$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, for $|r|<1$.

↳ However, in the case of the proposed series, $r=-2$, which is outside the range of convergence ($|r|<1$). This means that the formula does not apply, and the sum is not well-defined.

Euler's incorrect application of the geometric series sum formula led him to the sum $\frac{1}{3}$.

Exercise D

An observation: $\sum 2^k a_{2^k}$ sums the terms a_n at indices 2^k, but multiplies them by 2^k, creating a doubling effect.

Consider now the sum of terms between powers of 2:

$S_k = \sum_{n=2^k}^{2^{k+1}-1} a_n$. — Since (a_n) is monotone decreasing, we have:
$a_{2^k} \geq a_n \geq a_{2^{k+1}-1}$ for $2^k \leq n \leq 2^{k+1}-1$.

We can now estimate S_k:

$(2^{k+1}-2^k)a_{2^{k+1}-1} \leq S_k \leq (2^{k+1}-2^k)a_{2^k}$.

⇓ Now since $2^{k+1}-2^k = 2^k$, we obtain:

$2^k a_{2^{k+1}-1} \leq S_k \leq 2^k a_{2^k}$. We have the comparison we need, but we express it as sums:

$$\sum_{k=0}^{\infty} 2^k a_{2^{k+1}-1} \leq \sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k}.$$

↳ We know that if $\sum a_n$ converges the right-hand series must also converge. Conversely, if $\sum 2^k a_{2^k}$ converges, then $\sum a_n$ must also converge.

Thus, the two series converge if and only if each one of them converges.

Exercise E

From exercise D, we established that $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges and vice versa.

Applying this to the series in exercise E, we set:

$a_n = \frac{1}{n^p}$. Thus, the second series in our inequality becomes $2^k a_{2^k} = 2^k \cdot \frac{1}{(2^k)^p} \equiv 2^k \cdot 2^{-kp} = 2^{k(1-p)}$

↳ We can observe that the geometric series $\sum_{k=0}^{\infty} 2^{k(1-p)}$ has the common ratio $r = 2^{1-p}$.

We also know from calculus that $\sum r^k$ converges if and only if $|r|<1$, which gives: $|2^{1-p}|<1$.

↳ We are given that $p>0$, so we know that $2^{1-p}<1$, and a log of both sides gives $1-p<0$. Thus, $p>1$.

Given that the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is equivalent to the convergence of $\sum_{k=0}^{\infty} 2^{k(1-p)}$ (from exercise D) and, having established that the latter converges if and only if $p>1$, we can state that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p>1$.

Exercise L

Part a.

The root test states that for a series $\sum a_n$, we examine $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$ and, if:

- $L<1$, the series converges absolutely
- $L>1$, the series diverges
- $L=1$, the test is inconclusive, so another method must be used. This is the series in the question, where $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

We can try implementing a p-series test which states that $\sum \frac{1}{n^p}$ converges if $p>1$.

↳ Take $\sum_{n=1}^{\infty} \frac{1}{n^2}$. In this case, $p=2$, so the series converges. Let's see if this can be applied to the series from the question:

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}}$. Then, $\sqrt[n]{n^{-2}} = n^{\frac{-2}{n}}$ ← this tends to zero as $n\to\infty$. Thus, $\lim_{n\to\infty} n^{-2/n} = 1$, which is the same as the given series, so the test works.

The convergent series is thus $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Part b.

Take the classic harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ seen in the course so far.

↳ We know that, according to the p-series test, $\sum \frac{1}{n^p}$ diverges if $p\leq 1$. Here, $\frac{1}{n^1}$ implies $p=1$, implying that the series diverges. We can verify this:

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} \Rightarrow \sqrt[n]{n^{-1}} = n^{-1/n}$ which also tends to zero as $n\to\infty$. Then: $\lim_{n\to\infty} n^{-1/n} = 1$.

Thus, the condition is satisfied and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

2. Question 2

Part a.

The given identity is $\sum_{k=1}^{n} x_k y_k = s_n y_{n+1} + \sum_{k=1}^{n} s_k (y_k - y_{k+1})$.

Now, since s_n is bounded, there exists a number M so that $|s_n| \leq M$ for all n. Additionally, since y_k is monotonically decreasing and converges to zero we can state that:

$\lim_{n\to\infty} s_n y_{n+1} = 0$.

Onto the second term, we can note that $(y_k - y_{k+1})$ is positive and forms a telescoping sum. Then, remembering that s_n is bounded, we can write: $|s_k(y_k - y_{k+1})| \leq M|y_k - y_{k+1}|$.

↳ The comparison test tells us that, since $\sum(y_k - y_{k+1})$ is a telescoping sum and converges, so does $\sum s_k(y_k - y_{k+1})$

Now since both terms on the right side of the given identity converge, the left side $\sum x_k y_k$ converges as well, supporting the Dirichlet test.

Part b

The alternating series test states that if two conditions are met, that is, if:
1) y_k is monotonically decreasing and converges to zero, and
2) $x_k = (-1)^k$,

then $\sum(-1)^k y_k$ converges.

Let's think about this with regard to the Dirichlet test.

↳ First, the partial sums of $x_k = (-1)^k$ are bounded: $s_n = \sum_{k=1}^{n}(-1)^k$ (since we know that this oscillates).

Second, the sequence (y_k) is monotonically decreasing and tends to zero.

Both conditions of the alternating series test satisfy the Dirichlet test, so the former is a 'special' case of the latter.

3. Section 4.1

Exercise G

The hint states that if $\vec{x}$ makes an angle α with the positive y-axis, then $\vec{x} = (\|\vec{x}\| \cos\alpha, \|\vec{y}\| \sin\alpha)$. Similarly, if y makes an angle β with the positive x-axis then $\vec{y} = (\|\vec{y}\| \cos\beta, \|\vec{y}\| \sin\beta)$.

↳ The angle between x and y is then $\theta = \beta - \alpha$.

We can apply the standard dot product definition to $\vec{x}$ and $\vec{y}$:

$$\begin{aligned}
\vec{x}\cdot\vec{y} &= (\|\vec{x}\|\cos\alpha, \|\vec{x}\|\sin\alpha)\cdot(\|\vec{y}\|\cos\beta, \|y\|\sin\beta).\\
&= \|\vec{x}\|\|\vec{y}\|\cos\alpha\cos\beta + \|\vec{x}\|\|\vec{y}\|\sin\alpha\sin\beta. \quad \text{factoring-out } \|\vec{x}\|\|\vec{y}\|\\
&= \|\vec{x}\|\|\vec{y}\|(\cos\alpha\cos\beta + \sin\alpha\sin\beta). \quad \text{applying } \cos(\beta-\alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\\
&= \|\vec{x}\|\|\vec{y}\|\cos(\beta-\alpha)\\
&= \|\vec{x}\|\|\vec{y}\|\cos\theta.
\end{aligned}$$

We have successfully proven that the vector dot product $\langle\vec{x},\vec{y}\rangle$ is equal to $\|\vec{x}\|\|\vec{y}\|\cos\theta$.

Exercise H

Part a.

Take the norm squared identity: $\|x+y\|^2 = \langle\vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle$.

↳ We can expand the right side as: $\langle\vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle = \langle\vec{x},\vec{x}\rangle + 2\langle\vec{x},\vec{y}\rangle + \langle\vec{y},\vec{y}\rangle$.
We know that $\langle\vec{x},\vec{x}\rangle = \|\vec{x}\|^2$ and $\langle\vec{y},\vec{y}\rangle = \|y\|^2$. Then, reexpressing the above expansion;

$\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + 2(\vec{x},\vec{y}) + \|\vec{y}\|^2$. We have shown in exercise G that $\langle\vec{x},\vec{y}\rangle = \|\vec{x}\|\|\vec{y}\|\cos\theta$. Thus:
$= \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\|\cos\theta + \|\vec{y}\|$.

↳ Arriving at the cosine law for vectors proves it.

Part b.

Consider $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + 2(\vec{x},\vec{y}) + \|\vec{y}\|^2$, and $\|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 - 2(\vec{x},\vec{y}) + \|\vec{y}\|^2$.
Adding these two yields:
$$\|\vec{x}+\vec{y}\| + \|\vec{x}-\vec{y}\| = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$$

Dividing by 2: $\dfrac{\|\vec{x}+\vec{y}\|^2+\|\vec{x}-\vec{y}\|^2}{2} = \|\vec{x}\|^2 + \|\vec{y}\|^2$.

↳ We can see that adding the squared norm equations isolates $\|\vec{x}\|^2 + \|\vec{y}\|^2$.

Let's move on to the subtraction of the norms to see if we can arrive at an expression with $\langle\vec{x},\vec{y}\rangle$:

$$\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\| = (\|\vec{x}\|^2 + 2(\vec{x},\vec{y}) + \|\vec{y}\|^2) - (\|\vec{x}\|^2 - 2(\vec{x},\vec{y}) + \|\vec{y}\|^2).$$
$$= \cancel{\|\vec{x}\|^2} + 2\langle\vec{x},\vec{y}\rangle + \cancel{\|\vec{y}\|^2} - \cancel{\|\vec{x}\|^2} + 2\langle\vec{x},\vec{y}\rangle - \cancel{\|y\|^2}.$$
$$= 4\langle\vec{x},\vec{y}\rangle$$

↳ Thus, we can write: $\langle\vec{x},\vec{y}\rangle = \dfrac{\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2}{4}$

We have now successfully shown that the dot product can be expressed purely in terms of the norms of related vectors.

Question 4

Let's first negate absolute convergence. Note that a series a_n is absolutely convergent if $\sum|a_n|$ converges. In our case, we can observe that the series $\sum_{n\in\mathbb{Z}} \frac{1}{x-n}$ behaves very comparably to a harmonic series for sufficiently large n.

↳ We know that the harmonic series $\sum \frac{1}{|n|}$ diverges. Now, by the comparison test, since $\frac{1}{|x-n|} \geq \frac{1}{|n|}$ for large n, the absolute value of the given series diverges. Thus, the given series does not converge absolutely.

Let's now check for conditional convergence. We can rewrite the series by pairing the terms symmetrically about x:

$\sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{x+n}\right)$. This reorganisation is useful because, for large n, the two terms are nearly equal in magnitude but since they have opposite signs, they cancel out.

Now each component of the new series can have its denominates approximated by a first-order Taylor expansion:

1) $\dfrac{1}{x-n} = \dfrac{1}{-n(1-\frac{x}{n})} \approx \dfrac{-1}{n}\left(1+\frac{x}{n}\right) = -\frac{1}{n} - \frac{x}{n^2} + O(n^{-3})$

2) $\dfrac{1}{x+n} = \dfrac{1}{n(1+\frac{x}{n})} \approx \dfrac{1}{n} - \dfrac{x}{n^2} + O(n^{-3})$

↳ Adding these two approximations: $\dfrac{1}{x-n} + \dfrac{1}{x+n} = \left(-\frac{1}{n} - \frac{x}{n^2}\right) + \left(\frac{1}{n} - \frac{x}{n^2}\right) \approx -\dfrac{2x}{n^2} + O(n^{-3})$.

This result is important because it demonstrates that, at large n, the dominant behaviour of the sum is akin to a convergent p-series:

$\sum \frac{1}{n^2}$. Since $\sum 1/n^2$ converges $(p = 2 > 1)$, and since the sum in the question series is approximated by a comparable $-\frac{2x}{n^2}$, the bipartite series we constructed, i.e. $\sum_{n=1}^{\infty}\left(\frac{1}{x-n} + \frac{1}{x+n}\right)$, also converges by the comparison test.

We have initially established that the series does not converge absolutely, but we just showed that the series itself does converge. Thus, the series has to be conditionally convergent.