5.1 Exercises G and H

G. Suppose that $f:\mathbb{R}^n \to \mathbb{R}$ is continuous. If there are $\mathbf{x} \in \mathbb{R}^n$ and $C \in \mathbb{R}$ such that $f(\mathbf{x}) < C$, then prove that there is $r > 0$ such that for all $\mathbf{y} \in B_r(\mathbf{x})$, $f(\mathbf{y}) < C$.

H. Suppose that functions f, g, h mapping $S \subset \mathbb{R}^n$ into $\mathbb{R}$ satisfy $f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$ for $\mathbf{x} \in S$. Suppose that $\mathbf{c}$ is a limit point of S and $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{c}} h(\mathbf{x}) = L$. Show that $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = L$.

5.2 Exercise A

A. Let A be a subset of $\mathbb{R}^n$. Show that the characteristic function χ_A is continuous on the interior of A and of its complement A', but is discontinuous on the boundary $\partial A = \overline{A} \cap \overline{A'}$.

5.3 Exercises K and N

K. Show that $f:\mathbb{R}^n \to \mathbb{R}^m$ is continuous iff $f^{-1}(C)$ is closed for every closed set $C \subset \mathbb{R}^m$.

N. Suppose that $f:\mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f(u+v) = f(u) + f(v) \quad \text{for all} \quad u, v \in \mathbb{R}.$$

(a) Prove that $f(mx) = mf(x)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$. HINT: Use induction for $m \ge 1$.
(b) Prove that $f(x) = cx$ for all $x \in \mathbb{Q}$, where $c = f(1)$. HINT: Use (a) to solve for $f(p/q)$.
(c) Use (b) to prove that if f is continuous on $\mathbb{R}$, then $f(x) = cx$ for all $x \in \mathbb{R}$.

5.4 Exercises E and J

5.4.3. THEOREM. *Let C be a compact subset of $\mathbb{R}^n$, and let f be a continuous function from C into $\mathbb{R}^m$. Then the image set $f(C)$ is compact.*

PROOF. Let $(\mathbf{y}_k)_{k=1}^\infty$ be a sequence in $f(C)$. We must find a subsequence converging to a point in the image. First choose points $\mathbf{x}_k$ in C such that $\mathbf{y}_k = f(\mathbf{x}_k)$. Now $(\mathbf{x}_k)_{k=1}^\infty$ is a sequence in the compact set C. Therefore, there is a subsequence $(\mathbf{x}_{k_i})$ that converges to some $\mathbf{c}$ in C. By the continuity of f,

$$\lim_{i\to\infty} \mathbf{y}_{k_i} = \lim_{i\to\infty} f(\mathbf{x}_{k_i}) = f\big(\lim_{i\to\infty} \mathbf{x}_{k_i}\big) = f(\mathbf{c}).$$

Thus $(\mathbf{y}_{k_i})$ converges to $f(\mathbf{c}) \in f(C)$, showing that $f(C)$ is compact. ■

This immediately yields a result often used (without proof) in calculus.

5.3.1. THEOREM. *For a function f mapping $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$, the following are equivalent:*

(1) *f is continuous on S.*
(2) *For every convergent sequence $(\mathbf{x}_k)_{k=1}^{\infty}$ with $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$ in S, $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{a})$.*
(3) *For every open set U in $\mathbb{R}^m$, the set $f^{-1}(U) = \{\mathbf{x} \in S : f(\mathbf{x}) \in U\}$ is open in S.*

E. Show that f mapping a *compact* set $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is continuous iff its graph $G(f) = \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in S\}$ is compact. HINT: ($\Rightarrow$) use Theorem 5.4.3. ($\Leftarrow$) use Theorem 5.3.1 (2).

J. Let A be a compact subset of $\mathbb{R}^n$. Show that for any point $\mathbf{x} \in \mathbb{R}^n$, there is a closest point $\mathbf{a}$ in A to $\mathbf{x}$; i.e., $\mathbf{a} \in A$ satisfies $\|\mathbf{x} - \mathbf{a}\| \leq \|\mathbf{x} - \mathbf{b}\|$ for all $\mathbf{b} \in A$. HINT: Define a useful continuous function on A.

Question 5

5. Show that a uniformly continuous function preserves Cauchy sequences. In other words, show that, if $f\colon S \to \mathbb{R}^m$ is uniformly continuous on some set $S \subset \mathbb{R}^n$, and (x_n) is a Cauchy sequence in S, then $(f(x_n))$ is a Cauchy sequence. Use this to prove that a function $f\colon (a, b) \to \mathbb{R}$ is uniformly continuous if and only if it is possible to define values $f(a)$ and $f(b)$ so that the extended function $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

Question 6

6. Suppose that f is a function from the circle $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ onto $\mathbb{R}$. Prove that f cannot be continuous. Hint: Create a continuous function from the interval $[0, 2\pi]$ to the circle.

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MAT336 Homework 4

1. Section 5.1

Exercise G

Since f is continuous at x, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\|y - x\| < \delta \implies |f(y) - f(x)| < \varepsilon$.

We want $f(y) < C$. Since we know that $f(x) < C$ we can set $\varepsilon = C - f(x)$.

↳ Because $f(x) < C$, we can write that. Now, if $f(y)$ stays within ε of $f(x)$, then $f(y)$ must remain below C. We can thus apply continuity;

Continuity at x implies that there exists a $\delta > 0$ such that $\|y - x\| < \delta \implies |f(y) - f(x)| < \varepsilon$.

↓ plugging in our constructed ε:

$|f(y) - f(x)| < C - f(x)$.

↳ From $|f(y) - f(x)|$, now, we get 2 inequalities:

$-(C - f(x)) < f(y) - f(x) < C - f(x)$.

Focusing on the upper bound, this becomes:

$f(y) - f(x) < C - f(x) \implies f(y) < C$, which is what we need.

↳ We have shown that if we pick $r = \delta$, then for all $|y - x| < r$, it follows that $f(y) < C$.

Exercise H

$f(x) \leq g(x) + h(x)$. ⟶ sub $h(x)$ ⟶ $f(x) - h(x) \leq g(x)$.

↳ g is at least as large as $f(x) - h(x)$. We know that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = 0$. Then, by the properties of limits, we can say:

$\lim_{x \to c} (f(x) - h(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} h(x) = L - 0 = L$.

Now, since $f(x) - h(x) \leq g(x)$, it follows that $\liminf_{x \to c} (f(x) - h(x)) \leq \liminf_{x \to c} g(x)$.

↳ But we already know that $\lim_{x \to c} (f(x) - h(x)) = L$. Hence, $L = \lim_{x \to c} (f(x) - h(x)) \leq \liminf_{x \to c} g(x)$.

↳ In other words, $g(x)$ cannot be consistently smaller than $f(x) - h(x)$ in the limit, because $f(x) - h(x) \to L$.

We can conclude that if the limit $\lim_{x \to c} g(x)$ exists, then $\lim_{x \to c} g(x) \geq L$.

↳ Say that it is only known that the limit inferior of $g(x)$ exists. Still, then, $\liminf_{x \to c} g(x) \geq L$, which is enough to determine that $g(x)$ does not dip below L.

2. Section 5.2

Exercise A

We know from the course that a point $x \in \operatorname{int}(A)$ means that there exists some radius $r > 0$ s.t. the open ball $B_r(x)$ is completely contained in A; $B_r(x) \subset A$.

Because every $y \in B_r(x)$ lies in A, we have $\chi_A(y) = 1$ for all $y \in B_r(x)$. In other words, χ_A is constant $(=1)$ in the entirety of $B_r(x)$.

Now, a function that is constant on a neighborhood around x is necessarily continuous at x. Formally, for any $\epsilon > 0$, picking $\delta = r$, we can say that for all y with $\|y - x\| < \delta$,

$$|\chi_A(y) - \chi_A(x)| = |1 - 1| = 0 < \epsilon.$$

↳ We can thus tell that χ_A is continuous at x. Additionally, since x was an arbitrary point in $\mathrm{int}(A)$, χ_A is continuous on the entire interior of A.

In terms of the continuity on $\mathrm{int}(A^c)$, a point $x \in \mathrm{int}(A^c)$ means that there exists some radius $r > 0$ s.t. $B_r(x) \subset A^c$. We can restate this as:

$$B_r(x) \cap A = \phi.$$

↳ Now because every $y \in B_r(x)$ lies outside of A, we have that $\chi_A(y) = 0$ for all $y \in B_r(x)$. So, χ_A is constant on $B_r(x)$.

From here we can follow a similar reasoning as above;

$$|\chi_A(y) - \chi_A(x)| = |0 - 0| = 0 < \epsilon.$$

↳ Then, since x was an arbitrary point in $\mathrm{int}(A^c)$, χ_A is continuous on the entire interior of A^c.

Lastly, we have to address discontinuity at ∂A.

↳ We know from lecture that x is on the boundary ∂A if every neighborhood of x intersects both A and A^c.

Because the neighborhood (ball) $B_r(x)$ meets both A and A^c, there have to exist points y arbitrarily close to x where $\chi_A = 1$ (these are in A), as well as points z arbitrarily close to x where $\chi_A(z) = 0$ (these are in A^c).

Now for χ_A to be continuous at x, we have to make $\chi_A(y)$ arbitrarily close to $\chi_A(x)$ by choosing a y close to x.

↳ Where we run into a conflict is that, if $x \in A$, then $\chi_A = 1$; but no matter how small the ball around x, there are points in that ball for which $\chi_A = 0$. Conversely, if $x \notin A$, then $\chi_A(x) = 0$; but again, no matter how small the ball around x, there are points in the ball for which $\chi_A = 1$.

↳ In both cases, we can find points in every neighborhood around x wherein χ_A differs from $\chi_A(x)$. Hence, we cannot make $\chi_A(y)$ close to $\chi_A(x)$ for all y around x – it violates the ϵ–δ continuity condition.

↳ Thus, χ_A is discontinuous at every boundary point $x \in \partial A$.

3. Section 5.3

Exercise K

We will prove this scenario by first assuming that f is continuous.
We can also note that the complement of a closed set C in $\mathbb{R}^n$ is an open set, say $U = C^c$. We then know that:

$$f^{-1}(U) = f^{-1}(C^c) \text{ is open in } \mathbb{R}.$$

We can also clear up that the preimage of the complement is the complement of the preimage, i.e.:

$$f^{-1}(C^c) \iff (f^{-1}(C))^c.$$

↳ Because $f^{-1}(C^c)$ is open, its complement $(f^{-1}(C))$ must be closed. Hence, $f^{-1}(C)$ is closed for every closed set C.

We now have to prove the other direction; that is, we want our conclusion to be that f is continuous.

↳ Assuming that $f^{-1}(C)$ is closed for every closed set C, we need to show that the preimage of every open set is open.

Start by letting $V \subset \mathbb{R}^m$ be open. Then, its complement $V^c = C$ is closed. By the properties of sets, $f^{-1}(C)$ has to be closed in $\mathbb{R}^n$. Hence, its complement $(f^{-1}(C))^c$ is open in $\mathbb{R}^n$;

$$(f^{-1}(C))^c = f^{-1}(C^c) = f^{-1}(V).$$

↳ We can now conclude continuity. Since the preimage of every open set V is open, f is continuous.

We have shown the equivalence that f being continuous implies $f^{-1}(C)$ is closed for every closed set in C, and viceversa.

Exercise N

Part a

At the base case $(m=1)$, $f(1\cdot x) = f(x)$, and the statement $f(1\cdot x) = 1\cdot f(x)$ is trivial.

Now for $m \geq 2$:

Assume that we can manipulate $f((m-1)x) \iff (m-1)f(x)$. Then, via the provided "functional" equation;

$$f(mx) = f((m-1)x + x) = f((m-1)x) + f(x).$$

The induction hypothesis in this case is $f((m-1)x) = (m-1)f(x)$. Therefore;

$$f(mx) = (m-1)f(x) + f(x) = mf(x)$$

↳ So our induction proof is complete for all positive integers m.

Now for $m=0$:

Setting $u=0$, we get $f(0\cdot x) = f(0) = 0 = 0\cdot f(x)$.

Now for $m<0$:

Let $m=-k$ where $k>0$. Then, $f(mx) = f(-kx)$.
↳ We have the additive property in $\mathbb{R}$, i.e. $f(-x) = -f(x)$. Then;

$f(-kx) = -f(kx)$. We also already showed for $m>0$ that $f(kx) = kf(x)$. Thus;

$f(-kx) = -kf(x)$. Since $m=-kx$, it follows that $f(mx) = mf(x)$.

We have shown for all integer ranges that $f(mx) = mf(x)$.

Part b

We have just established that $f(n) = nf(1)$ for all $n \in \mathbb{Z}$. We can let $c = f(1)$ and say that $f(n) = cn$ for all $n \in \mathbb{Z}$.

For positive denominators in $\mathbb{Q}$:

Consider an integer m, a positive integer n, and $f(\frac{m}{n})$. We can say that $nf(\frac{m}{n}) = f(n\cdot\frac{m}{n}) = f(m)$.
↳ But in our case, $f(m) = cm$. Hence:

$$nf\left(\frac{m}{n}\right) = cm \implies f\left(\frac{m}{n}\right) = \frac{m}{n}c. \text{ Thus, } f\left(\frac{m}{n}\right) = c\frac{m}{n}.$$

For negative denominators in $\mathbb{Q}$:

If $n<0$, we can say that $\frac{m}{n} = \frac{-m}{-n}$ and apply the same logic as above. We also know from part a that $f(-x) = -f(x)$.

We have shown that for all rational $x = \frac{m}{n}$ with $n \neq 0$, $f(x) = cx$.

Part c

We know from week 1 that $\mathbb{Q}$ is dense in $\mathbb{R}$, and that for $x \in \mathbb{R}$, we can pick a sequence of rational numbers $\{q_n\}$ such that $q_n \to x$ as $n \to \infty$.

↳ To apply continuity to this, we can say that, knowing $f(q_n) = cq_n$ for all n, and given continuity of f on $\mathbb{R}$;

$$\lim_{n\to\infty} f(q_n) = f\left(\lim_{n\to\infty} q_n\right) = f(x).$$

↳ But $\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} cq_n = cx$.

This allows us to conclude that $f(x) = x$ for all $x \in \mathbb{R}$.

4. Section 5.4

Exercise F

We have to construct a 2-way proof.

1.) Proving that if f is continuous, then $G(f)$ is compact.
2.) Proving that if $G(f)$ is compact, then f is continuous.

Proof 1.

Let's take the map $G: S \to \mathbb{R}^{n+m}$, $G(x) = (x, f(x))$. Here, G is continuous because both x and $f(x)$ are continuous.

↳ Now, it is commonly known that the continuous image of a compact set is compact. Hence, $G(f)$ is compact in $\mathbb{R}^{n+m}$.

Proof 2

We can start off by noting that in $\mathbb{R}^{n+m}$, any compact set is closed. So, $G(f)$ is a closed subset of $\mathbb{R}^{n+m}$.

To prove f is continuous on S, we can think in terms of sequences and say that f is continuous if and only if whenever $x_k \to x$ in S, it implies $f(x_k) \to f(x)$ in $\mathbb{R}^m$.

↳ Let's assume a sequence $x_k \to x$ in S. Because $x_k \to x$, and because S is contained in some closed/bounded region, we can anticipate that $\{(x_k, f(x_k))\}$ has a convergent subsequence, and we can think about its limit.

If $(x_k, f(x_k)) \to (x, y)$ in $\mathbb{R}^{n+m}$, the limit point must lie in $G(f)$ because $G(f)$ is closed.

↳ But $(x, y) \in G(f)$ means that $y = f(x)$.

↳ We can thus conclude that $(x_k, f(x_k)) \to (x, f(x))$ in $\mathbb{R}^{n+m}$. This implies that $f(x_k) \to f(x)$ in $\mathbb{R}^m$. Thus, $f(x)$ is continuous at x. Since x was arbitrary in S, f is continuous on all of S.

We have shown both initial directions, i.e. that f being continuous on S implies that $G(f)$ is compact in $\mathbb{R}^{n+m}$, and viceversa.

Exercise J

Let's define a function as per the hint: $g: A \to \mathbb{R}$, $g(a) = \|x - a\|$.

↳ The construction of $g(a)$ allows us to think about it as the distance from a to the fixed point x. Our goal is to show that g attains its minimum value somewhere in A. That is, that there exists a $a^* \in A$ so that:

$$g(a^*) = \min_{a\in A} g(a).$$

Let's first verify that the distance function is continuous. We could apply the triangle inequality (it is often applicable in 'distance' functions).

↳ By the reverse triangle inequality;

$$\big|\, \|x - a\| - \|x - a'\| \,\big| \leq \|a - a'\|.$$

Now, let $\varepsilon > 0$ be determined so that $\delta = \varepsilon$ is chosen. Then, if $\|a - a'\| < \delta$, it follows that:

$$|g(a) - g(a')| = \big|\, \|x - a\| - \|x - a'\| \,\big| \leq \|a - a'\| < \varepsilon.$$

↳ This shows that for every $\varepsilon > 0$, there exists a $\delta = \varepsilon$ such that $\|a - a'\| < \delta \implies |g(a) - g(a')| < \varepsilon$. Hence, g is continuous on A.

We now need to find a way to show that the minimum of the function is attained/reached in A.

↳ First, take the infimum of g on A. Let

$$m = \inf_{a \in A} g(a)$$

By definition of the infimum then, there exists a sequence $\{a_k\} \subset A$ s.t.:

$$g(a_k) \to m \text{ as } k \to \infty.$$

Now because A is compact, every sequence in A has a convergent subsequence. Hence, there exists a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to some point $a^* \in A$.

Additionally, since g is continuous, we have $g(a_{k_j}) \to g(a^*)$.

↳ But, since $g(a_{k_j})$ is a subsequence of $g(a_k)$ and we know $g(a_k) \to m$, it follows that $g(a^*) \to m$. Therefore, the minimum value m is attained at the point $a^* \in A$.

Question 5

Part 1: Showing that a uniformly continuous function preserves Cauchy sequences

Let's first clear up the definition of uniform continuity. f being uniformly continuous on S means that:

For all $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \varepsilon$, for all $x, y \in S$.

We can also recall the definition of a Cauchy sequence:

For all $\delta > 0$, there exists an $N \in \mathbb{N}$ s.t. $\|x_n - x_m\| < \delta$.

We can interact/combine the two definitions:
- Given any $\varepsilon > 0$, pick the corresponding δ from the uniform continuity of f.
- Since (x_n) is Cauchy, there is an N s.t. for all $n, m > N$, $\|x_n - x_m\| < \delta$.
- By uniform continuity, $\|f(x_n) - f(x_m)\| \leq \varepsilon$ for all $n, m > N$.

↳ We can notice that $(f(x_n))$ satisfies the Cauchy criterion in $\mathbb{R}$. Therefore, $(f(x_n))$ is a Cauchy sequence.

Part 2: Characterizing Uniform Continuity on (a,b) via a Continuous Extension to $[a,b]$.

The claim that this part of the question makes is that a function $f: (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b) if & only if we can define values $f(a)$ and $f(b)$ so that the 'extended function'

$$\tilde{f}: [a,b] \to \mathbb{R}, \quad \tilde{f}(x) = \begin{cases} f(x), & x \in (a,b), \\ f(a), & x = a, \\ f(b), & x = b. \end{cases}$$

is continuous on the closed interval $[a,b]$.

It is probably best to prove each direction separately.

1) That if f is uniformly continuous on (a,b), then f extends continuously to $[a,b]$.

Take any sequence (x_n) in (a,b) with $x_n \to a$. Then, (x_n) is a Cauchy sequence (since all convergent sequences in $\mathbb{R}$ are Cauchy).

In Part 1 we have shown that since f is uniformly continuous, $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$. Hence, $(f(x_n))$ converges to some $L \in \mathbb{R}$.

If we were to take another sequence (y_n) with $y_n \to a$, you could similarly argue that $(f(y_n))$ converges to some L'.
↳ Uniform continuity then forces $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$. Thus, the limit is well-defined.

We can define $f(a) := L$. This ensures continuity at $x = a$, because $\lim_{x\to a^+} f(x) = f(a)$.

↳ By the same argument, if (x_n) is any sequence in (a,b) that converges to b, then the sequence $(f(x_n))$ is Cauchy in $\mathbb{R}$. Let this limit be M.

Then, defining $f(b) := M$, we have $\lim_{x\to b^-} f(x) = f(b)$.

Extending f to $\tilde{f}$ on $[a,b]$ by these values at the endpoints makes $\tilde{f}$ continuous on $[a,b]$. Hence, we have shown the first direction. Now onto the second:

2) That if f extends to a continuous f on $[a,b]$, then f is uniformly continuous on (a,b).

We know that $[a,b]$ is compact in $\mathbb{R}$. The Heine-Cantor theorem, then, states that a continuous function on a compact set is uniformly continuous on that set.

However, since $\tilde{f}$ is uniformly continuous on the entire interval $[a,b]$, its restriction to (a,b) must also be uniformly continuous. But this restriction-in-question is f as per the question setup, so f is uniformly continuous on (a,b).

We have succinctly completed the proof in the other direction.

By demonstrating that if f extends continuously to $[a,b]$ then f is uniformly continuous on (a,b), and conversely that uniform continuity on (a,b) ensures the existence of continuous limits at a and b (allowing for a continuous extension to $[a,b]$), we have fully established the equivalence stated in Question 5.

Question 6

We can tell that the set S^1 is compact in $\mathbb{R}^2$. Then, we can define a continuous map from $[0,2\pi]$ to S^1 as per the hint:

$$g : [0, 2\pi] \to S^1,\ g(\theta) = (\cos\theta, \sin\theta).$$

↳ The map is continuous and $g([0,2\pi]) = S^1$.

Now if f were continuous and onto $\mathbb{R}$, then $f \circ g : [0,2\pi] \to \mathbb{R}$ would be a continuous surjection from the compact interval $[0,2\pi]$ onto $\mathbb{R}$.

But we arrive at a contradiction, i.e. that the image of a compact set must be compact.
↳ In our case, $\mathbb{R}$ is not compact! Thus, $(f\circ g)([0,2\pi])$ cannot possibly be all of $\mathbb{R}$. This contradicts our earlier assumption that f is onto $\mathbb{R}$.

The contradiction means that no continuous 'onto' map $f : S^1 \to \mathbb{R}$ can exist.