1. 6.1. Exercise F

F. Prove the product rule for functions f and g on $[a,b]$ that are differentiable at x_0.
HINT: $f(x_0+h)g(x_0+h)-f(x_0)g(x_0) = \big(f(x_0+h)-f(x_0)\big)g(x_0+h)+f(x_0)\big(g(x_0+h)-g(x_0)\big)$.

2. 6.2 Exercise D

D. Suppose that f is C^3 on (a,b), and f has four zeros in (a,b). Show that $f^{(3)}$ has a zero.

3. 6.3 Exercise H

H. (a) Show that if $f \geq 0$ is Riemann integrable on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

(b) If f and g are integrable on $[a,b]$ and $f(x) \leq g(x)$, prove that $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
(c) Show that $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

4. 6.4 Exercises C.(a), E

C. (a) Prove the **Mean Value Theorem for Integrals**: If f is a continuous function on $[a,b]$, then there is a point $c \in (a,b)$ such that $\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)$.

E. Let f be a continuous function on $\mathbb{R}$, and suppose that $b(x)$ is a C^1 function. Define $G(x) = \int_a^{b(x)} f(t)\,dt$. Compute $G'(x)$. HINT: Let $F(x) = \int_a^x f(t)\,dt$ and note that $G(x) = F(b(x))$.

5.

A set $A \subset [a,b]$ is said to have *content zero* if, for every $\epsilon > 0$, there exists a finite collection of open intervals $\{O_1, O_2, \ldots, O_n\}$ whose union contains A and whose lengths sum to ϵ or less. Letting $|O_j|$ denote the length of the interval O_j, we can write this as

$$A \subset \bigcup_{j=1}^{n} O_j \qquad \text{and} \qquad \sum_{j=1}^{n} |O_j| \leq \epsilon. \tag{1}$$

Suppose that $f\colon [a,b] \to \mathbb{R}$ is bounded. Show that, if the set of discontinuous points of f has content zero, then f is integrable. Prove that the Cantor set C defined in §4.4.8 has content zero. Use this fact, together with the fact that C is closed, to show that the function

$$h(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}, \tag{2}$$

is integrable, and find the value of the integral.

6.

Consider Thomae's function

$$t(x) = \begin{cases} 1 & \text{if } x = 0, \\ 1/n & \text{if } x = m/n \in \mathbb{Q}, \text{ with } m \in \mathbb{Z} \setminus \{0\},\ n \in \mathbb{N}, \text{ and } \gcd(m,n) = 1, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases} \tag{3}$$

Prove that t is Riemann integrable on the interval $[0,1]$. Hint: Set $\epsilon > 0$ and let $N \in \mathbb{N}$ be large enough so that $1/N < \epsilon/2$. Define the set $D_N = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{1}{N}, \frac{2}{N}, \ldots \frac{N-1}{N}\}$. Then, observe that $t(x) \geq 1/N$ whenever $x \in D_N$, and that $t(x) < 1/N$ otherwise. Create a partition P with $\text{mesh}(P) < \epsilon/(2|D_N|)$ such that $D_N \cap P = \{0, 1\}$.

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MAT336 Homework 5

1. Section 6.1

Exercise F

Let f and g be functions defined on an interval (a,b) that are differentiable at $x_0 \in (a,b)$. We want to prove that the derivative of their product $h(x) = f(x)g(x)$ at x_0 is:

$$h'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

↳ We know the definition of the derivative from class, and that we need to evaluate the equivalent expression:

$$h'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}$$

⇓ We can rewrite the numerator...

$$f(x_0+h)g(x_0+h) - f(x_0)g(x_0) \iff [f(x_0+h) - f(x_0)]g(x_0+h) + f(x_0)[g(x_0+h) - g(x_0)].$$

Thus we can reexpress the entire derivative expression as:

$$\frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) + f(x_0)\frac{g(x_0+h) - g(x_0)}{h}.$$

We have allowed that f and g are differentiable at x_0. Thus, we can comfortably say the following:

- $\frac{f(x_0+h) - f(x_0)}{h} \to f'(x_0)$ as $h \to 0$.
- $\frac{g(x_0+h) - g(x_0)}{h} \to g'(x_0)$ as $h \to 0$
- $g(x_0+h) \to g(x_0)$ as $h \to 0$.

Combining these three in the limit we defined gives:

$$\lim_{h \to 0}\left[\frac{f(x_0+h) - f(x_0)}{h} g(x_0+h) + f(x_0)\frac{g(x_0+h) - g(x_0)}{h}\right]$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

We have effectively shown that $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$, which is a proof of the product rule.

2. Section 6.2

Exercise D

It seems like we're working with Rolle's theorem. Let f be a function in C^2 on the interval (a,b). Also suppose that f has 4 separate zero roots in (a,b). We can classify these as:

$$x_1 < x_2 < x_3 < x_4, \text{ all in } (a,b).$$

↳ In this question, we seek to demonstrate that there exists some point say, d, s.t. $f''(d) = 0$.

We can begin by applying Rolle's theorem to f. We are permitted to do this because

$f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$ as per our setup. Thus, with f taking the same endpoint value;

- There exists a $c_1 \in (x_1, x_2)$ such that $f'(c_1) = 0$,
- There exists a $c_2 \in (x_2, x_3)$ such that $f'(c_2) = 0$, and so on for (x_3, x_4).

↳ We can deduce using this that f' has 3 distinct zeros; $c_1 < c_2 < c_3$ in (a,b). We can thus also apply Rolle's Theorem to f' on the sub-intervals (c_1, c_2) and (c_2, c_3):

- There must exist a $d_1 \in (c_1, c_2)$ s.t. $f''(d_1) = 0$, and
- By the same logic, there must exist a $d_2 \in (c_2, c_3)$ s.t. $f''(d_2) = 0$.

↳ We can use this finding, in turn, to determine that the function $f''(d)$ has at least 2 distinct zeros; $d_1, d_2 \in (a,b)$ and, in particular, there must be at least one d in (a,b) with $f(d)=0$. We have completed the proof at this point.

3. Section 6.3

Exercise H

Evidently, we are dealing with a proof of 3 simple properties of the Riemann Integral (RI)

Part a.)

A function is Riemann integrable on $[a,b]$ if for every $\varepsilon>0$, there is a partition "P" of $[a,b]$ s.t. the Riemann sums $S(f,P)$ are arbitrarily close to the same limit (the value that the integral evaluates to).

Additionally, if $f(x)\geq 0$ for all $x\in[a,b]$, then every Riemann sum is non-negative.

↳ The integral, as the limit of these non-negative sums, must itself be non-negative.

⇓ We can argue the same with 'lower-sum' notation;

$$L(f,P) = \sum_{i=1}^{n}\left(\inf_{x\in(x_{i-1},x_i)} f(x)\right)(x_i - x_{i-1}).$$

↳ Here, since $f\geq 0$, each infimum is ≥ 0, so $L(f,P)\geq 0$ as well. The integral is the supremum of all lower sums, hence $\int_a^b f(x)\,dx > 0$.

Part b

We should analyze f versus g.

↳ Since $f(x)\leq g(x)$, we know that $g(x)-f(x)\geq 0$. We also know from part a above that if a function is non-negative its integral is nonnegative too. Thus;

$$\int_a^b [g(x)-f(x)]\,dx \geq 0. \xrightarrow{\text{distributing}} \equiv \left[\int_a^b g(x)\,dx - \int_a^b f(x)\,dx\right]\geq 0 \equiv \left[\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx\right]$$

We have proved the exact expression from the question.

Part c

We have the basic inequality that we can apply here; $-f(x)\leq f(x)\leq |f(x)|$.

↳ Now, since f and $|f|$ can be integrated, and given that $-|f(x)|\leq f(x)\leq |f(x)|$ holds for all $x\in[a,b]$, we can invoke our insight from part b, that is the monotonicity of integrals. We can specifically use it to deduce:

$$\int_a^b (-|f(x)|)\,dx \equiv -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

↳ This implies $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$, which is the exact statement from the question.

4. Section 6.4

Exercise C.(a.)

We can think of this question as the statement: "prove that there is a point c where the constant average value of f on $[a,b]$ equals the actual value of f."

Now, since f is continuous on the closed interval $[a,b]$, we can apply the extreme value theorem and say that f attains its minimum value (m) and maximum value (M) somewhere in $[a,b]$.

Hence:

$m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$.

↳ Therefore, for all $x \in [a,b]$, $m \leq f(x) \leq M$, which is intuitively trivial.
Now, integrating this inequality in the bounds of x gives:

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx.$$

We can reexpress the above, knowing that $\int_a^b m\,dx = m(b-a)$ and $\int_a^b M = M(b-a)$:

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \iff m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

↳ Now to interpret this, we can note that the intermediate value property tells us that since f is continuous on $[a,b]$ and takes all values between m and M, there must be some c in the same range so that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

↳ We get the exact expression for the Mean Value Theorem that we were looking for.

Exercise E

We will show differentiability and compute $G'(x)$ using the FTC and the Chain Rule, respectively.

Let $F(u) = \int_a^u f(t)\,dt$. By the FTC, then, F is differentiable with $F'(u) = f(u)$ (f is continuous).

↳ Applying this insight to G:

$$G(x) = \int_a^{b(x)} f(t)\,dt = F(b(x)).$$

Now, since $G(x) = F(b(x)) \cdot b'(x)$, we can apply the Chain rule for composite functions:

$$G'(x) = F'(b(x)) \cdot b'(x).$$

↳ But, $F'(u) = f(u)$ as per the FTC outline we made. Hence;

$$G'(x) = f(b(x))\,b'(x).$$

↳ Thus, whenever b is a C^1 function and f is continuous, G is differentiable and it's derivative is equal to:

$$G'(x) = f(b(x))\,b'(x).$$

5. Question 5 Provided

Let's make a couple observations about the Cantor set, that we have first seen in lecture, in order to better understand the setup and requirements of the question:

The Cantor set is constructed by repeatedly removing the open middle thirds from the interval $[0,1]$. It is:

1) Closed (contains all of its limit points).
2) Uncountable (it is countably infinite).
3) Content zero (can be covered by intervals whose total length is arbitrarily small. In lecture we said that the set has "area zero").

Henceforth, we will denote the Cantor set as C.

Question 5 introduces the piecewise function $h(x) = \begin{cases} 1, & \text{if } x \in C, \\ 0, & \text{if } x \notin C. \end{cases}$, with the interval $[0,1]$.

↳ We can instantly tell that, because $h(x)$ only takes on 0 or 1, $|h(x)|$ has to be smaller-than or equal to 1 for all x, so h is bounded.

Now, examining what happens at each of those points more closely:

1) At $x \in C$:

Every neighbourhood of x contains points not in C. At these points, h takes the value of 0, thereby jumpin from 1 at x to 0 when arbitrarily close to x. This means that h is discontinuous at each $x \in C$.

2) At $x \notin C$:

Here, x lies in one of the open intervals removed in the iterative process of C. There is a small open interval around x that is completely outside of C. In that neighbourhood, h is zero. So, h is continuous at every $x \notin C$.

↳ The key insight from this is that the set of discontinuities of h is precisely the Cantor set.

Now, since h has content/area zero, and this is synonymous to the set of discontinuities of h, we know from the property of integrability that h is Riemann-integrable.

We now just need to obtain $\int_0^1 h(x)\,dx$. In Riemann integration terms, and thinking about an area-based computation, having a value of 1 on a set of content zero means that no area whatsoever is represented by the integral.

↳ We can capture this idea formally, by making three sequential observations:

1) We can cover C by finitely-many intervals of arbitrarily-small lengths.
2) Outside of these small intervals, $h(x) = 0$.
3) Inside these small intervals, the contribution to the upper sum can be at most $1 \cdot \sum$ (interval lengths), but these lengths can be made arbitrarily close to length zero!

$\Downarrow$ Thus

$$\int_0^1 h(x)\,dx = 0.$$

⟵ This makes sense. If the lower sum is equal to zero and the upper sum can be made arbitrarily close to zero, the entire expression evaluates to zero.

↳ We have arrived at the value of the integral that the question asks for, and have shown that f is Riemann-integrable. We have also at the start laid out the conditions that make the Cantor set's "content"/area equal to zero.

6. Question 6 Provided

Fix $\varepsilon > 0$. Now, choose $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Also, let $D_N = \{0\} \cup \{\frac{m}{n} \in [0,1] : n < N, \gcd(m,n) = 1\}$.

↳ Note that D_N is finite and that $t(x) \geq 1/N$ iff $x \in D_N$, except for $x = 0$ where $t(0) = 1$.

Now, we can construct a partition P of $[0,1]$ with the following constraints/properties:

1) Each point of D_n except the endpoints is the midpoint of one of the subintervals of P.
2) The mesh of P is so small that the total length of all sub-intervals containing points of D_N is less than $\varepsilon/(2 \cdot \max(t))$.

Now, note that on sub-intervals that do not contain any point of D_N, the supremum of $t(x)$ is smaller than $\frac{1}{N}$. Conversely, on those that do contain a point of D_N, the supremum is no larger than one.

↳ But, importantly, the latter sub-intervals are made so short that their total contribution to the Riemann sum is still smaller than $\varepsilon/2$.

Consequently, the upper sum $U(P,t)$ is less than ε. Since the lower sum is always 0, we can use integral notation to denote:

$$\overline{\int_0^1} t\,dx = \inf_P U(P,t) = 0,$$ meaning t is Riemann-integrable on $[0,1]$ and $\int_0^1 t(x)\,dx = 0$.