1. Section 7.7. Exercise 7.a.

C. If M is a closed subspace of a Hilbert space H, define the **orthogonal complement** of M to be $M^{\perp} = \{x : \langle x, m\rangle = 0 \text{ for all } m \in M\}$.

(a) Show that every vector in H can be written uniquely as $x = m + y$, where $m \in M$ and $y \in M^{\perp}$. Moreover, $\|x\|^2 = \|m\|^2 + \|y\|^2$.

Hint: First, show that M is a Hilbert space. You can assume that H (and therefore M) is separable, and you can use without proof the fact that every Hilbert space has an orthonormal basis.

2. Section 7.7. Exercise D.

D. Let P be a projection on an inner product space V. Prove that the following are equivalent:

(a) P is an orthogonal projection.
(b) $\|v\|^2 = \|Pv\|^2 + \|v - Pv\|^2$ for all $v \in V$.
(c) $\|Pv\| \leq \|v\|$ for all $v \in V$.
(d) $\langle Pv, w\rangle = \langle v, Pw\rangle$ for all $v, w \in V$.

HINT: For (c) $\Longrightarrow$ (d), show that *not* (d) implies there are vectors $v = Pv$ and $w = (I - P)w$ such that $\langle v, w\rangle > 0$. Compute $\|v - tw\|^2 - \|v\|^2$ for small $t > 0$. For (d) $\Longrightarrow$ (a), show that $\langle Pv, (I - P)w\rangle = 0$.

Hint: Notice that (a) = (b) follows from Proposi tion 7.5.10, and that (b) = (c) immediately. For (c) = (d) and (d) = (a), see the hint in the exercise.

3. Dini's Theorem Question

(**Dini's theorem**). Suppose that $f_n \to f$ on a compact set $K \subset \mathbb{R}^n$. Suppose that, at each $x \in K$, $(f_n(x))$ is an increasing sequence. Show that, if f_n and f are continuous on K, then (f_n) converges to f uniformly.

(a) Let $g_n = f - f_n$. Translate the hypothesis of the theorem above into statements about the sequence (g_n).

(b) Prove that, if $K_1 \supset K_2 \supset \cdots$ is a sequence of compact sets in $\mathbb{R}^n$ and $\cap_{n=1}^{\infty} K_n = \emptyset$, then $K_m = \emptyset$ for some $m \in \mathbb{N}$. Hint: Suppose that $K_n \neq \emptyset$ for all $n \in \mathbb{N}$. Define a sequence (x_n) where $x_n \in K_n$ for each $n \in \mathbb{N}$, and use this sequence to prove that $\cap_{n=1}^{\infty} K_n \neq \emptyset$.

(c) Let $\epsilon > 0$ and define $K_n = \{x \in K : g_n(x) \geq \epsilon\}$. Show that $K_1 \supset K_2 \supset \cdots$. Use this sequence of sets to prove the theorem. Hint: Use part (b) to prove that $g_n \to 0$ uniformly.

4. Section 8.6. Exercise A.

A. Use $f_n(x) = x^n$ on $[0, 1]$ to show that $B = \{f \in C[0, 1] : \|f\| \leq 1\}$ is not compact.

5. Provided Question

Prove that the sequence $f_n = \sin(nx)/n$ is equicontinuous on $[0, 1]$.

Filip Zaleski 04.04.2025

MAT336 Homework 6

1. Section 7.7

Exercise C.a.

We know that M is a closed subspace of the Hilbert space H, so it must also be complete under the inner product $\langle\cdot,\cdot\rangle$ restricted to M. Thus, M is itself a Hilbert space.

Now because H is separable, it has a countable orthonormal basis. Having verified that $M \subset H$ is a Hilbert space itself, and taking from the question's construction its separability, it also has an orthonormal basis.

↳ We could now try extending the orthonormal basis $\{e_n\}$ of M to obtain an orthonormal basis for all of H:

$$\underbrace{\underbrace{\{e_n\}_{n=1}^{\infty}}_{\text{spans } M} \subset \underbrace{\{e_n\}_{n=1}^{\infty} \cup \{f_k\}_{k=1}^{\infty}}_{\text{spans } H}}_{\text{together still orthonormal}}$$

We can now write any $x \in H$ as an infinite sum of projections onto the basis vectors:

$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n + \sum_{k=1}^{\infty}\langle x, f_k\rangle f_k$, where $\langle x, e_n\rangle$ represents the coordinate of x in the direction of e_n.

↳ We can split this sum into 2 parts:

$$\underbrace{m}_{\text{in } M} = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n, \qquad \underbrace{y}_{\text{in } M^{\perp}} = \sum_{k=1}^{\infty}\langle x, f_k\rangle f_k.$$

We still need to verify that the above is the case; that is, that m is in M and that y is in the orthogonal complement of M:

- We know that e_n vectors lie in M, forming its orthonormal basis.
- We also know that a finite/countable linear combination of vectors that are all inside M is still in M.

↳ We can thus say that the sum $\sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ lies in the closure of the span of $\{e_n\}$, which itself is M. Now since M is complete, the limit of this sum is inside M too. Thus, $m \in M$.

Now onto $y \in M^{\perp}$:

- We know that each f_k is orthogonal to each e_n, i.e. $\langle f_k, e_n\rangle = 0$.
- We also know that in order for $M^{\perp}$ to include y, y must satisfy $\langle y, m'\rangle = 0$ for all $m' \in M$.

↳ Every $m' \in M$ can be approximated by a finite linear combination of the basis vectors e_n of M. Hence, we can restrict our focus to just the condition $\langle y, e_n\rangle = 0$ for all n.

↳ Since we know that $y = \sum_k \langle x, f_k\rangle f_k$, and given each f_k is orthogonal to each e_n, we know:

$\langle f_k, e_n\rangle = 0$ ← which, by the linearity of the inner product, implies;

$\langle y, e_n\rangle = \left\langle \sum_k \langle x, f_k\rangle f_k, e_n\right\rangle = \sum_k \langle x, f_k\rangle\langle f_k, e_n\rangle = \sum_k \langle x, f_k\rangle 0 = 0.$

↳ Thus, y is orthonormal to all the e_n's (thus to any $m' \in M$), so $y' \in M^{\perp}$.

What's left now is to verify that the decomposition $x = m + y$ is not only possible, but also unique.

↳ Let's construct two sub-decompositions:

1) $x = m_1 + y_1$, and
2) $x = m_2 + y_2$.
$\Big\}$ with $m_1, m_2 \in M$ and $y_1, y_2 \in M^\perp$.

⇓ Subtracting them:

$x - x = (m_1 + y_1) - (m_2 + y_2) = (m_1 - m_2) + (y_1 - y_2) = 0$. Thus $m_1 - m_2 = -(y_1 - y_2)$.

But,

- $m_1 - m_2$ is a vector in M
- $y_1 - y_2$ is a vector in $M^\perp$

↳ The problem with this is that the long equation expresses one vector both as something in M and something in $M^\perp$. The only vector that satisfies this is the zero vector. Thus;

$m_1 - m_2 = 0$, $y_1 - y_2 = 0$, so $m_1 = m_2$, $y_1 = y_2$.

↳ This proves uniqueness

In terms of the norm identity, the fact that m and y are orthogonal (i.e. $\langle m, y \rangle = 0$) gives us a sort-of Pythagorean formula of the form:

$$\|x\|^2 = \langle x, x \rangle = \langle m + y, m + y \rangle = \langle m, m \rangle + \langle m, y \rangle + \langle y, m \rangle + \langle y, y \rangle.$$

↳ Since $\langle m, y \rangle = \langle y, m \rangle = 0$, we can say:

$$\|x\|^2 = \|m\|^2 + \|y\|^2.$$

↳ This completes the proof: we have shown that every vector $x \in H$ uniquely decomposes into an element of M, plus an element of $M^\perp$, and that their lengths obey the convenient Pythagorean relationship.

Exercise D

We have the following four conditions:

a) P is an orthogonal projection.
b) $\|v\|^2 = \|Pv\|^2 + \|(v - Pv)\|^2$ for all v.
c) $\|Pv\|^2 \leq \|v\|$ for all v.
d) $\langle Pv, w \rangle = \langle v, Pw \rangle$ for all v, w.

↳ We will iteratively show that they are equivalent, beginning with (a) <=> (b). This obeys the sequential approach to proving several concurrent properties that Professor Serkh performed in lecture.

Step 1: That (a) <=> (b)

First, that (a) => (b):

If P is an orthogonal projection, then each v splits as $v = Pv + (v - Pv)$ and $\langle Pv, v - Pv \rangle = 0$.

From the properties of orthogonality, we also know: $\|v\|^2 = \|Pv + (v - Pv)\|^2 = \|Pv\| + \|v - Pv\|^2$.

↳ We have arrived at statement (b).

Now the other direction; that (b) => (a):

If it is true that $\|v\|^2 = \|Pv\|^2 + \|v - Pv\|^2$ for every $v \in V$, then $\langle Pv, v - Pv \rangle = 0$. In other words, Pv is always orthogonal to $(I - P)v$. With $P^2 = P$, this perfectly meets the definition of an orthogonal projection.

We have proved both directions, so onto step 2.

Step 2 : That (b) $\Rightarrow$ (c).

Condition (b) states $\|v\|^2 = \|Pv\|^2 + \|(v - Pv)\|^2$ for all v. Taking square roots of both sides grants $\|v\| \geq \|Pv\|$.

↳ This is precisely equal to condition (c). Onto the next step.

Step 3 : That (c) $\Rightarrow$ (d).

We need to show that if P satisfies $P^2 = P$ and $\|Px\| \leq \|x\|$ for all x, then $\langle Pv, w\rangle = \langle v, Pw\rangle$ for all v, w.

↳ This can be reinterpreted as showing that $\langle Pv, (I-P)w\rangle = 0$. Here, as per the given hint, we can show that if $\langle Pv, (I-P)w\rangle \neq 0$, then we can have a vector x for which $\|Px\| > \|x\|$, which contradicts condition (c).

We can, then, prove by contradiction. Assume that $\langle Pv, (I-P)w\rangle \neq 0$. We can also assume that $\langle Pv, (I-P)w\rangle > 0$.

↳ Then, define $v' = Pv$ and $w' = (I-P)w$.

↳ Then, v' is in the image of P while w' is in the kernel of P (since $Pw' = P(I-P)w = Pw - P^2w = Pw - Pw = 0$).

Now, consider the set of vectors $x(t) = v' - tw'$ for $t \in \mathbb{R}$. Then,

$P(x(t)) = P(v') - tP(w') = v' - t \cdot 0 = v'$. So, $\|P(x(t))\| = \|v'\|$ does not rely on t.

At the same time, $\|x(t)\|^2 = \|v' - tw'\|^2 = \|v'\|^2 - 2t\langle v', w'\rangle + t^2\|w'\|^2$.

↳ Because $\langle v', w'\rangle > 0$, for small enough $t > 0$, the middle term in this expression will shrink the value of $\|x(t)\|^2$ more than the term t^2 can compensate for. In particular, for sufficiently small $t > 0$;

$\|x(t)\|^2 < \|v'\|^2 \Rightarrow \|x(t)\| < \|v'\|$.

However, recall that $\|Px(t)\| = \|v'\|$. So, for the small t values;

$\|Px(t)\| = \|x(t)\|$. ⟵ This contradicts property (c), which says that $\|Px(t)\| \leq \|x(t)\|$ for <u>all</u> $x(t)$.

↳ Hence, the assumption $\langle Pv, (I-P)w\rangle = 0$ was impossible, mandating:

$\langle Pv, (I-P)w\rangle = 0$ for all v, w $\Rightarrow \langle Pv, w\rangle = \langle v, Pw\rangle$.

↳ Thus, (c) forces (d), and we have proved by contradiction.

Step 4 : That (d) $\Rightarrow$ (a).

If it is the case that $\langle Pv, w\rangle = \langle v, Pw\rangle$ for all v, w, then P is a self-adjoint operator – i.e. $P = P^*$. This fact interacted with the projection property $P^2 = P$ means that P must be an orthogonal projection.

↳ In particular, self-adjointness combined with $P^2 = P$ ensures that range(P) is orthogonal to range$(I-P)$. Reexpressed, this means that, for each v;

$\langle Pv, v - Pv\rangle = 0 \Rightarrow \|v\|^2 = \|Pv\|^2 + \|v - Pv\|^2$.

↳ This indicates that P is orthogonal.

Altogether, we have shown:

$$\begin{tikzcd} (a) \arrow[r, bend left] & (b) \arrow[r, bend left] & (c) \arrow[r, bend left] & (d) \arrow[lll, bend left] \end{tikzcd}$$

Our proof is thus complete.

3. A Provided Question : Dini's Theorem

Part a.)

Because $f_n \to f$ is pointwise, we have $f_n(x) \leq f(x)$ for all n large than some threshold. Thus;

$g_n(x) = f(x) - f_n(x) \geq 0$ for large n.

Then, the fact that f_n is increasing in n at each x means that $f_1(x) \leq f_2(x) \leq \ldots$

↳ Substituting $g_n = f - f_n$ implies that $g_1(x) \geq g_2(x) \geq \ldots$

↳ We can thus say that $\{g_n\}$ is a decreasing sequence of nonnegative continuous functions on K, converging pointwise to zero.

We can now infer that Dini's Theorem is equivalent to showing that a pointwise-decreasing sequence $\{g_n\}$ of nonnegative continuous functions on a compact set 'K' must converge to zero uniformly. This is so if the convergence is pointwise.

Part b

We will prove by contradiction and use the hint.

Suppose that all K_n are nonempty. Then, we can pick $x_n \in K_n$ for each n.

↳ Because each K_n is compact, Bolzano-Weierstrass tells us that any sequence drawn from a compact set must have a convergent subsequence.

We also know from the nested intervals lemma that, given $K_1 \supseteq K_2 \supseteq \ldots$, every tail of the sequence $\{x_n\}$ stays in all the earlier K_j's as well. The limit of that convergent subsequence must lie in every K_n, i.e. in $\bigcap_n K_n$.

↳ The problem is that we assumed at the start that $\bigcap_n K_n$ was empty; so there's a contradiction. We can thus conclude that some K_n must already be empty, so that the sequence cannot remain nonempty indefinitely.

Part c

We can now apply our findings from part b to show that the convergence $g_n \to 0$ is uniform.

Let's use an error-term-based approach and fix $\varepsilon > 0$ so that we can define the set K_n which represents the set of points in K where $g_n(x)$ is at least ε.

$K_n = \{x \in K : g_n(x) \geq \varepsilon\}$.

Now since $\{g_n\}$ is decreasing with n, we have that $g_{n+1}(x) \leq g_n(x)$.

↳ Thus, if x satisfies $g_{n+1}(x) \geq \varepsilon$, then automatically $g_n(x) \geq g_{n+1}(x) \geq \varepsilon$. This means that $x \in K_{n+1}$ would imply $x \in K_n$. Hence, $K_{n+1} \subseteq K_n$.

Thus, we can state the general case, that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$

Let's see what other observations we can make about K_n.

↳ First, if $g_n(x) \to 0$ for each $x \in K$, then for every $\varepsilon > 0$, eventually $g_n(x) < \varepsilon$. Thus, x cannot stay in all of the K_n's – belonging to K_n requires $g_n(x) > \varepsilon$. We can in turn deduce that each $x \in K$ is absent from K_n once n is large enough.

↳ We can thus state that $\bigcap_{n=1}^{\infty} K_n = \emptyset$.

At this point, we know that each K_n is a closed set and, using the above two insights we just made about K_n, we can declare that because K is compact, if all K_n were nonempty and nested, their intersection could not be empty.

↳ The fact that the intersection is empty necessitates at least one K_N being empty.

↳ But then, for any $n \geq N$, we know that $g_n(x) \leq g_N(x) < \varepsilon$, which holds for all $x \in K$.

Hence;

$$\sup_{x \in K} g_n(x) \leq \varepsilon \text{ for all } n \geq N.$$

Thus, $g_n(x) \longrightarrow 0$ uniformly on K.

↳ This discovery allows us to draw the final insight: since $g_n = f - f_n$, the statement $g_n \to 0$ uniformly is equivalent to $f_n \to f$. This completes the proof of Dini's theorem.

4. Section 8.6

Exercise 4.

The question defines $f_n(x) = x^n$ on $[0,1]$ and $B = \{f \in C[0,1] : \|f\| \leq 1\}$.

↳ We can instantly confirm that each f_n is continuous on $[0,1]$ and $\|f_n\| = \sup_{x \in [0,1]} |x^n| = 1$.

Thus, $f_n \in B$ for all n.

We can also tell from the construction of $f_n(x)$ that f_n has a pointwise nature:

$$x^n \longrightarrow \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

The limit function is thus e.g. $$p(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x = 1. \end{cases}$$

The limit function is not continuous. This then means that f_n cannot converge uniformly to any continuous function. The justification for this is three-fold:

1) If f_n were to converge in $\|\cdot\|_\infty$, the limit would have to be continuous (i.e. some $f \in [0,1]$).

2) Being pointwise, however, the sequence implies a limit that is discontinuous at $x = 1$.

3) There cannot, then, exist a function $f \in C[0,1]$ that satisfies $\lim_{n \to \infty} \|f_n - f\|_\infty = 0$.

↳ These observations allow us to state that $\{f_n\}$ has no convergent subsequence in the supremum norm.

We know from lecture about compact sets in a metric space that they must be sequentially compact, i.e. they must have convergent subsequences whose limits also lie in the set.

↳ Now because $\{f_n\} \subset B$ has no uniformly-convergent subsequence in B, B fails sequential compactness and is thus not compact.

5. A Provided Question

The backdrop to the question:

A family of functions $\{f_n\}$ on $[0,1]$ is equicontinuous if, for every $\varepsilon > 0$, there is a $\delta > 0$ s.t. for all n and all $(x,y) \in [0,1]$ with $|x - y| < \delta$, one can state:

$$|f_n(x) - f_n(y)| < \varepsilon.$$

In our setting, f_n is given by:

$$f_n(x) = \frac{\sin(nx)}{n}.$$

↳ Showing that $f_n(x)$ is equicontinuous uniformly in n seems like it could involve the Mean Value Theorem. We are working with a norm involving x and y, after all.

By the Chain Rule, $f_n'(x) = \frac{d}{dx}[\sin(\frac{nx}{n})] = \frac{n\cos(nx)}{n} = \cos(nx)$.

↳ We know that $|\cos(nx)| \leq 1$ for all n, x.

Now, for any $x, y \in [0,1]$, there is some point z between x and y such that $f_n(x) - f_n(y) = f_n'(z)(x-y)$.

↳ Reexpressed: $|f_n(x) - f_n(y)| = |f_n'(z)||x-y| \leq 1 \cdot |x-y|$.

↳ We can tell that the upper bound does not rely on n.

↳ Hence, we can assertain that for every $\varepsilon > 0$, if $|x-y| < \delta = \varepsilon$, then:

$|f_n(x) - f_n(y)| \leq |x-y| < \varepsilon$.

↳ This works simultaneously for all n. Thus, $\{f_n\}$ is equicontinuous.